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Comments on anomalies and charges of toric-quiver duals

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Abstract: We obtain a simple expression for the triangle 't Hooft anomalies in quiver gauge theories that are dual to toric Sasaki-Einstein manifolds. We utilize the result and simplify considerably the proof concerning the equivalence of a-maximization and Z-minimization. We also resolve the ambiguity in defining the flavor charges in quiver gauge theories. We then compare coefficients of the triangle anomalies with coefficients of the current-current correlators and find perfect agreement.

Keywords: AdS-CFT Correspondence, Global Symmetries, Supersymmetric gauge theory, Gauge-gravity correspondence.

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1. Introduction

In recent years, a large number of new examples of $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence [1] have been constructed and studied extensively. IIB string theory on $\mathrm{AdS}_5 \times Y$ preserves $\mathcal{N} = 2$ supersymmetry (8 supercharges) when Y is a Sasaki-Einstein (SE) manifold [2 – 5]. Soon after the discovery of new SE metrics [6, 7], it was realized that many of the SE manifolds are toric [8 – 10]. When Y is toric, most geometric quantities such as its volume can be computed without knowledge of the explicit metric [9]. The toric description also helped identifying $\mathcal{N} = 1$ superconformal gauge theory duals [11 – 14], the quiver gauge theories. Using new techniques to analyze quiver gauge theories, very detailed checks have been made for toric-quiver dual pairs [15]–[25].

One such issue concerns identifying the correct R-symmetry at the conformal fixed point. The superconformal $U(1)_R$ symmetry is in general a nontrivial linear combination of all nonanomalous global $U(1)$ symmetries. In gauge theory dual, it was found in [26] that maximizing a-function determines uniquely the correct combination. Denoting the global charges as Q_I, the definition of a as a function of the trial R-charge contains the triangle 't Hooft anomaly, whose coefficient is given by[1]

$$C_{IJK} = \mathrm{Tr}(Q_I Q_J Q_K). \tag{1.1}$$

The rule of a-maximiation in $\mathcal{N} = 1$ supersymmetric gauge theory and its geometric dual have played a crucial role throughout the development [27]–[31]. The conserved currents J_I associated with the charges Q_I are mapped to $U(1)$ gauge fields A^I in supergravity via AdS/CFT correspondence. Then the anomaly coefficient C_{IJK} is encoded [32] as the coefficient of the Chern-Simons term in the five-dimensional gauged supergravity action

$$S_{CS} \sim \int C_{IJK}\, A^I \wedge F^J \wedge F^K. \tag{1.2}$$

The anomaly coefficients C_{IJK} is also suggested intimately related to the coefficients τ_{IJ} of the two-point correlators among conserved currents via τ_{RR} minimization [33].

While the gauge theory expression for C_{IJK} (1.1) is now available from [12, 28], the supergravity expression in terms of geometric data on SE manifold has been lacking so far (see, however, the paragraphs below). On the contrary, the expression for τ_{IJ} is known in supergravity [30] but not in the gauge theory. To make a connection between C_{IJK} and τ_{IJ} as suggested in [33], one thus needs a more geometric understanding of C_{IJK}. In fact, from the supergravity viewpoint, the connection ought to exist since τ_{IJ} and C_{IJK} are both derivable from an underlying prepotential $\mathcal{F}$ [34].

In this work, we report progress in comparing global charges and anomalies from gauge theory and those from supergravity. In particular, we identify the flavor charges in gauge theory unambiguously and use the identification to compare the expression for triangle 't Hooft anomalies in supergravity and gauge theory.

Our work begins in section 2 with a simple observation that the gauge theory result for the triangle 't Hooft anomaly coefficients as derived in [12, 28] is nothing but the area of a triangle connecting three vertices on the toric diagram:

$$C_{IJK} = \frac{1}{2}|\langle v_I, v_J, v_K\rangle|. \tag{1.3}$$

After deriving this formula, we illustrate its use by re-deriving the equivalence [28] of a-maximization and its geometric counterpart, Z-minimization [9]. Although our proof is similar to the original one [28], the use of (1.3) reduces the amount of needed computation considerably. We also resolve the ambiguity in defining the non-R 'flavor' charges in the gauge theory so as to facilitate the comparison with supergravity results.

Clearly, the next logical step is to compute C_{IJK} in supergravity by performing perturbative Kaluza-Klein (KK) reduction up to cubic order. While we were making progress in that direction, ref. [40] appeared, in which a supergravity formula for C_{IJK} valid for any (not necessarily toric or Sasakian) Einstein manifold, as well as the gauge theory

[1]Throughout this paper, we work in the usual large N limit and suppress the dependence on N. It can be easily reinstated so that a is proportional to N^2, F_i^I is proportional to N, etc.

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result (1.3), were obtained. Section 3 of our paper is organized accordingly. After reviewing the linearized approximation to KK reduction and fixing the normalization of the charges, we show that the flavor charges computed in field theory in section 2 agrees perfectly with the supergravity result [30]. Finally, we make an explicit check of the relation $\tau_{IJ} = -3C_{RIJ}$ [33] using the result from [30, 40] and again find perfect agreement.

2. Toric quiver gauge theory side

It is by now well-known that the global $U(1)$ symmetries of a gauge theory with an SE dual are divided into two kinds. One is called baryon symmetry, and corresponds to $D3$-branes wrapping calibrated three-cycles of the SE manifold Y. The other is often called flavor symmetry and is associated with the isometry of Y. How the gauge fields for each $U(1)$ symmetry arise in the AdS_5 gauged supergravity will be reviewed in section 3.

In the toric case, Y has three isometries by definition, and the number of independent three-cycles are given by the toric data. Both symmetries are most efficiently described in the language of toric geometry, not only on the supergravity side but also in the quiver gauge theory. So, we shall begin with a quick review of well-known facts about the toric geometry of Y, mainly to establish our notations and summarize some results pertinent to discussion in later sections. See [8, 9] for more information on toric geometry in this context.

2.1 A short review of toric SE manifolds

It is useful to define the SE manifold Y in terms of the cone $X = C(Y)$ with the metric

$$ds_X^2 = dr^2 + r^2 ds_Y^2. \tag{2.1}$$

The manifold Y being Sasakian is equivalent to the cone X being Kähler. The Reeb Killing vector field defined as

$$K_R = \mathcal{I}\left(r\frac{\partial}{\partial r}\right), \tag{2.2}$$

where I denotes the complex structure on X, is translated to the R-symmetry of the field theory dual. The manifold Y is Sasaki-Einstein if X is Kähler and Ricci-flat, i.e., Calabi-Yau (CY). It is known that when Y is SE, it can be locally described as the $U(1)_R$ fibration over a Kähler-Einstein base B. The following relations will be useful when we prove some identities in section 3:[2]

$$\begin{aligned} ds_X^2 &= dr^2 + r^2((e^0)^2 + ds_B^2), \qquad e^0 \equiv \frac{1}{3}d\psi + \sigma, \qquad K_R = 3\frac{\partial}{\partial\psi}, \\ J_X &= r^2 J_B + r dr \wedge e^0, \qquad \Omega_X = e^{i\psi} r^2 \Omega_B \wedge (dr + ire^0), \\ R_{\mu\nu}^{(B)} &= 6g_{\mu\nu}^{(B)}, \qquad d\sigma = 2J_B, \qquad d\Omega_B = 3i\sigma \wedge \Omega_B. \end{aligned} \tag{2.3}$$

[2]Generically, B is an orbifold rather than a smooth manifold. Some of the proofs in section 3 involve integration by parts over B, hence they are not strictly valid. But, we expect that similar proofs will work with mild modifications.

In physics terminology, a toric cone X is conveniently described by the gauged linear sigma model (GLSM). For X, the GLSM takes a D-term Kähler quotient of $\{Z^I\} \in \mathbb{C}^d$ with respect to integer charges Q_a^I:

$$\sum_{I=1}^{d} Q_a^I |Z^I|^2 = 0, \qquad Z^I \sim e^{iQ_a^I\theta^a} Z^I \qquad (a = 1, \cdots, d-3), \tag{2.4}$$

leaving a three-dimensional complex cone. The CY condition sets $\sum_I Q_a^I = 0$ for each a.

Let $\{v^i\}$ $(i = 1, 2, 3)$ be a basis of the kernel of the map $Q_a : \mathbb{Z}^d \to \mathbb{Z}^{d-3}$, i.e., $Q_a^I v_I^i = 0$. One can regard v_I^i as d lattice vectors in $\mathbb{Z}^3$ and use them to parameterize $|Z^I|^2 = v_I \cdot y \equiv v_I^i y_i$ $(y \in \mathbb{R}^3)$. The allowed values of y form a polyhedral cone Δ defined by $\{v_I \cdot y \geq 0\}$ in $\mathbb{R}^3$. The cone X is then a fibration of three angles $\{\phi^i\}$ over the base Δ. Using the CY condition $\sum_I Q_a^I = 0$, one can set $v_I^1 = 1$ for all I, as this assignment satisfies $Q_a^I v_I^1 = 0$ automatically. We will always set $v_I^1 = 1$. The polygon drawn on $\mathbb{R}^2$ with the remaining components of v_I's is usually called the toric diagram.

By definition, a toric Y has three isometries $K_i = \partial/\partial\phi^i$. The Reeb vector K_R is in general a linear combination of them, $K_R = b^i K_i$. In [9], it was shown that the Reeb vector characterizes all the essential geometric properties of Y. The manifold Y is embedded in X as $Y = X \cap \{b \cdot y = 1/2\}$. Supersymmetric cycles of Y are given by $\Sigma^I = Y \cap \{v_I \cdot y = 0\}$.

The Reeb vector also determines a unique Sasakian metric on Y. The volume of Y is computable by summing over the volume of the supersymmetric cycles [9]:

$$\mathrm{Vol}(Y) = \frac{\pi^3}{b^1} \sum_I \frac{\langle v_{I-1}, v_I, v_{I+1}\rangle}{\langle b, v_{I-1}, v_I\rangle \langle b, v_I, v_{I+1}\rangle}. \tag{2.5}$$

Here, $\langle u, v, w\rangle$ denotes the determinant of the (3×3) matrix made out of vectors u, v, w. The CY condition on X fixes $b^1 = 3$. The metric of Y becomes Einstein at the minimum of $\mathrm{Vol}(Y)$ as b^2, b^3 are varied inside the polyhedral cone: $b \in \Delta$.

As explained in [13], when Y is simply-connected, which we assume for the rest of this paper, the homology group of Y is given by $H_3(Y, \mathbb{Z}) = \mathbb{Z}^{d-3}$. If $\{C^a\}$ $(a = 1, \cdots, d-3)$ form a basis of three-cycles of Y, it can be shown that $\Sigma^I = Q_a^I C^a$, where Q_a^I is precisely the GLSM data (2.4) of Y. The harmonic three-forms ω_a dual to C^a measure the baryon charges of Σ^I, so

$$B_a\left[\Sigma^I\right] = \int_{\Sigma^I} \omega_a = Q_a^I. \tag{2.6}$$

As one can see from the torus action in the GLSM description (2.4), the baryon charges Q_a^I and the flavor charges F_i^I together span $\mathbb{Z}^d$ (for simply connected Y). This means that the toric relation $Q_a^I v_I^i = 0$ can be extended to

$$\begin{pmatrix} {Q_a}^I \\ {F_i}^I \end{pmatrix} \begin{pmatrix} {u_I}^b & {v_I}^j \end{pmatrix} = \begin{pmatrix} \delta_a^b & 0 \\ 0 & \delta_i^j \end{pmatrix}, \tag{2.7}$$

for some integer-valued matrices F_i^I and u_I^b. One may want to interpret F_i^I as the i-th flavor charges of Σ^I, i.e., $F_i\left[\Sigma^I\right] = F_i^I$. However, even after choosing a fixed basis for v_I^i,

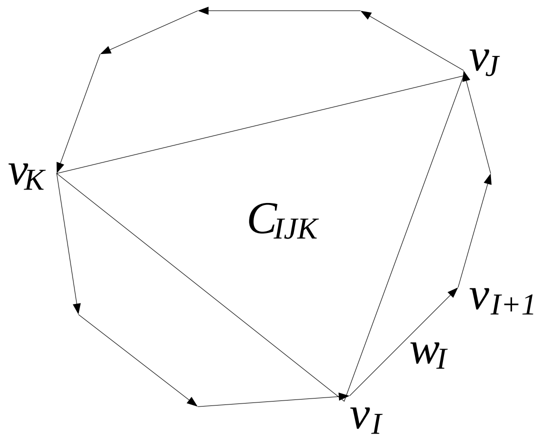

Figure 1: Triangle anomaly coefficient as the area of a triangle on the toric diagram.

the relation (2.7) does not fix F_i^I uniquely, as one may shift F_i^I and u_I^b by

$$F_i{}^I \to F_i{}^I + N_i{}^a Q_a{}^I, \qquad u_I{}^b \to u_I{}^b - v_I{}^i N_i{}^b. \tag{2.8}$$

This freedom is called the mixing ambiguity in the literature; flavor symmetry is unique up to mixing with baryon symmetries. This immediately poses a question: in comparing the gauge theory results with the supergravity results, how are the flavor charges on both sides to be identified? Later in this section, we will show that there is a unique, preferred choice of (non-integer) F_i^I which matches with the supergravity result.

2.2 Triangle anomaly from triangle area

We shall now derive a formula for the triangle 't Hooft anomaly of quiver gauge theories dual to Y. The formula states that the anomaly coefficient $C_{IJK} = \mathrm{Tr}(Q_I Q_J Q_K)$ is simply the area of the triangle connecting the three vertices $v_{I,J,K}$ on the toric diagram:

$$C_{IJK} = \frac{1}{2}|\langle v_I, v_J, v_K\rangle|. \tag{2.9}$$

The derivation of (2.9) is built upon some known features of the quiver gauge theories [28]:

1. The number of gauge group F is twice the area of the toric diagram.

2. Let $w_I \equiv (v_{I+1} - v_I)$ denote the edges of the toric diagram. Associated with each pair of edges (w_I, w_J), there are bifundamental chiral superfields Φ^r_{IJ} with the same charges (see below) and multiplicity given by $|\langle w_I, w_J\rangle| \equiv |w_I^2 w_J^3 - w_I^3 w_J^2|$.

See [28] and references therein for more details. The formula (2.9) is then derivable from the expression for the a-function for the quiver gauge theories.

An explicit expression for the a-function was given in [28]. First, a trial R-charge h^I is assigned to each vertex of the toric diagram subject to the constraint, $\sum_I h^I = 2$. The vertex v_I is associated to a $D3$-brane wrapped on the calibrated three-cycle Σ^I in Y through

$v_I \cdot y = 0$. Then, the R-charge of Φ_{IJ} is $R(\Phi_{IJ}) = \sum_{K=I+1}^{J} h^K$ or $R(\Phi_{IJ}) = \sum_{K=J+1}^{I} h^K$ depending on the sign of $\langle w_I, w_J\rangle$. The trial a-function is given by [28]

$$\begin{aligned}\frac{32}{9}a = C_{IJK}h^I h^J h^K &= F\left(\frac{1}{2}\sum h^I\right)^3 + \sum_{I<J}\langle w_I, w_J\rangle\left(\sum_{K=I+1}^{J} h^K - \frac{1}{2}\sum h^I\right)^3 \\ &\equiv Fx^3 + \sum_{I<J}\langle w_I, w_J\rangle\,(y_{IJ} - x)^3\,. \end{aligned} \tag{2.10}$$

The first term is the contribution of gaugini while the other terms account for the fermionic components of Φ^r_{IJ}. We replaced 1's appearing in the formula of [28] by $\frac{1}{2}\sum_I h^I$ using the constraint $\sum_I h^I = 2$ as we want to express a as a homogeneous cubic function of h^I's and read off the anomaly coefficients.

In the simplest case, $d = 3$, we can check (2.9) explicitly,

$$\begin{aligned}\frac{32}{9}a &= F(x^3 + (h^1 - x)^3 + (h^2 - x)^3 + (h^3 - x)^3) \\ &= 3Fh^1h^2h^3 = 6 \times \frac{1}{2}|\langle v_1, v_2, v_3\rangle| h^1h^2h^3, \end{aligned} \tag{2.11}$$

where we used $\langle w_1, w_2\rangle = \langle w_2, w_3\rangle = \langle w_3, w_1\rangle = |\langle v_1, v_2, v_3\rangle| = F$. Now, we proceed by induction. Assume the relation (2.9) holds for a toric diagram with d vertices, and then add another vertex v_{d+1}. We distinguish the objects for the new diagram by putting tilde above them.

$$\begin{aligned}\frac{32}{9}\tilde{a} &= \tilde{F}\tilde{x}^3 + \sum_{I<J}^{d+1}\langle \tilde{w}_I, \tilde{w}_J\rangle\,(\tilde{y}_{IJ} - \tilde{x})^3 \\ &= (F + \langle \tilde{w}_d, \tilde{w}_{d+1}\rangle)\left(x + \frac{1}{2}h^{d+1}\right)^3 + \sum_{I<J}^{d-1}\langle w_I, w_J\rangle\left(y_{IJ} - x - \frac{1}{2}h^{d+1}\right)^3 \\ &\quad + \sum_{I=1}^{d-1}\langle w_I, \tilde{w}_d\rangle\left(y_{Id} - x - \frac{1}{2}h^{d+1}\right)^3 + \sum_{I=1}^{d-1}\langle w_I, \tilde{w}_{d+1}\rangle\left(y_{Id} - x + \frac{1}{2}h^{d+1}\right)^3 \\ &\quad + \langle \tilde{w}_d, \tilde{w}_{d+1}\rangle\left(-x + \frac{1}{2}h^{d+1}\right)^3. \end{aligned} \tag{2.12}$$

By collecting terms with $(h^{d+1})^n$ $(n = 0, 1, 2, 3)$, one can show that (2.9) holds for all $d+1$ vertices. [3] The simplest one turns out to be the $(h^{d+1})^0$ term. Setting $h^{d+1} = 0$, we readily find

$$\tilde{a}|_{h^{d+1}=0} = a, \tag{2.13}$$

since the $\langle \tilde{w}_d, \tilde{w}_{d+1}\rangle$ terms cancel out and $\langle w_I, \tilde{w}_d\rangle + \langle w_I, \tilde{w}_{d+1}\rangle = \langle w_I, w_d\rangle$. In fact, we can use (2.13) to reverse the direction of the mathematical induction. That is, we can begin with $d > 3$ vertices and choose any three for which we want to compute C_{IJK}. Then (2.13) allows us to remove the rest of the vertices successively until we finally reach $d = 3$. The value of C_{IJK} does not depend on the other vertices.

[3] We thank Eunkyung Koh for carrying out this 'forward' proof completely.

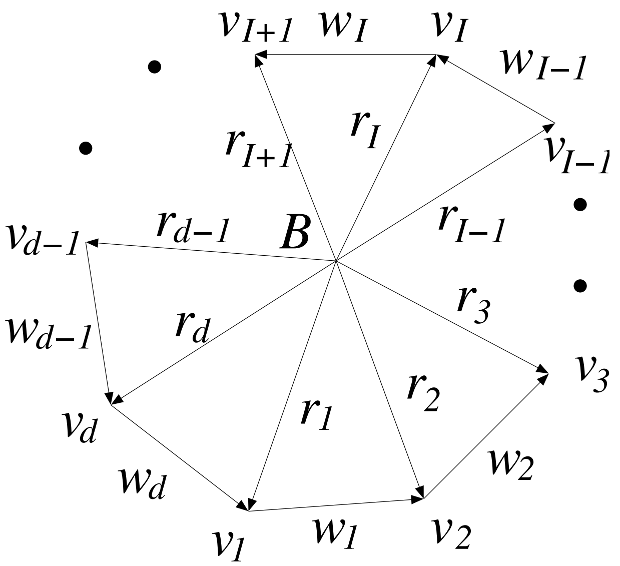

Figure 2: The Reeb vector as a point B inside the polygon [28].

2.3 Applications

To demonstrate the utility of the compact formula (2.9), we shall now apply it to rederive two known results.

First, let us show that the triangle 't Hooft anomaly of baryon symmetries always vanishes [28]: $\mathrm{Tr}\, B^3 = C_{IJK} B^I B^J B^K = 0$, where B^I is an arbitrary linear combination of the baryon charges only: $B^I = t^a Q^I_a$. For example, when $d = 4$,

$$\begin{aligned} \frac{1}{3} C_{IJK} B^I B^J B^K &= \langle B^1 v_1, B^2 v_2, B^3 v_3 \rangle + \langle 2,3,4 \rangle + \langle 3,4,1 \rangle + \langle 4,1,2 \rangle \\ &= \langle (1+2+3+4), 2, 3 \rangle + \langle (1+2+3+4), 4, 1 \rangle \\ &= 0. \end{aligned} \tag{2.14}$$

In the last step, we used the toric relation $\sum_I Q^I_a v^i_I = 0$. Similarly, for arbitrary d, vanishing of $\mathrm{Tr}\, B^3$ follows from $\sum_I B^I (B^J B^K \langle v_I, v_J, v_K \rangle) = 0$ (no sum over J, K). We relegate the general proof to appendix A.

Second, let us show the equivalence of a-maximization in a quiver gauge theory and Z-minimization of the dual toric SE manifold proposed in [9] and proven in [28]. Following [28], we parameterize the Reeb vector by $(b^1, b^2, b^3) = 3(1, x^2, x^3)$ and define

$$r_I = (x^2, x^3) - (v^2_I, v^3_I), \quad A_I = \langle r_I, w_I \rangle, \tag{2.15}$$

$$L^I(x^2, x^3) = \frac{\langle w_{I-1}, w_I \rangle}{A_{I-1} A_I}, \quad S = \sum_I L^I. \tag{2.16}$$

Then the results of [9] can be translated to the following forms of trial R-charges and a-function:

$$h^I_{MSY} \equiv \frac{2L^I}{S} \qquad \text{and} \qquad a_{MSY} = \frac{9}{32} \left(\frac{24}{S} \right). \tag{2.17}$$

In [28], it was shown that maximization of a_{CFT} with respect to trial R charges is equivalent to maximization of a_{MSY} with respect to the Reeb vector components (x^2, x^3). The first step of the proof asserts that the baryon charges decouple from the maximization process:

$$\mathrm{Tr}\, BR^2|_{h^I=h^I_{MSY}} = 0 \qquad \Longleftrightarrow \qquad C_{IJK}B^I L^J L^K = 0. \tag{2.18}$$

Then it remains to prove that the maximization process yields the same result. In fact, a_{CFT} and a_{MSY} are shown to be equal even before maximization:

$$a_{CFT}|_{h^I=h^I_{MSY}} = a_{MSY} \qquad \Longleftrightarrow \qquad C_{IJK}L^I L^J L^K = 3S^2. \tag{2.19}$$

A complete proof of these two steps were presented in the (rather long) appendix of [28].

Here we note that (2.9) offers a shorter and perhaps more intuitive proof. As we prove in the appendix, both of the above statements follow from a single lemma:

$$c_I \equiv C_{IJK}L^J L^K = 3S + \langle r_I, u\rangle, \tag{2.20}$$

where u is some vector independent of the label I. If the lemma is true, (2.18) follows from $\sum_I Q^I_a = 0 = \sum_I Q^I_a v_I$ and (2.19) from $\sum_I L^I r_I = 0$. The proof of the lemma is quite straightforward if we combine (2.9) with the original reasoning of [28]. See appendix B.

2.4 More on the flavor charges and decoupling

In gauge theory, we maximize the a-function

$$a = \frac{9}{32} C_{IJK}h^I h^J h^K, \tag{2.21}$$

subject to the constraint $\sum h^I = 2$. As the R-charge is a linear combination of baryon and flavor charges, we can write

$$h^I = t^a Q^I_a + s^i F^I_i. \tag{2.22}$$

In this new basis, the constraint means $s^1 = 2$, as one can see from the extended toric relation (2.7) and $v^1_I = 1$. In fact, s^i are related to the Reeb vector simply as $s^i = (2/3)b^i$. At this stage, as discussed in section 2.1, F^I_i is ambiguous. The values of t^a at the maximum of the a-function depend on the choice of F^I_i, while the values of s^i and the a-function do not.

As discussed less explicitly in [28], we can perform the maximization process in two steps.

$$\bar{a}(s,t) \equiv \frac{1}{3}C_{IJK}h^I h^J h^K = C_{iab}s^i t^a t^b + C_{ija}s^i s^j t^a + \frac{1}{3}C_{ijk}s^i s^j s^k \tag{2.23}$$

$$\equiv m_{ab}(s)t^a t^b + 2n_a(s)t^a + \frac{1}{3}C_{ijk}s^i s^j s^k. \tag{2.24}$$

This is a quadratic function of t^a, so maximization with respect to t^a is done trivially to give $\bar{t}^a(s) = -m^{ab}(s)n_b(s)$. Inserting it back to (2.22),

$$\bar{h}^I(s) = -Q^I_a m^{ab}(s)n_b(s) + F^I_i s^i, \tag{2.25}$$

$$\bar{a}(s) = -m^{ab}(s)n_a(s)n_b(s) + \frac{1}{3}C_{ijk}s^i s^j s^k. \tag{2.26}$$

The result discussed in the last subsection suggests the following identification:

$$h^I_{MSY}(x^2,x^3) = \bar{h}^I(s)|_{s=2(1,x^2,x^3)}, \qquad a_{MSY}(x^2,x^3) = \frac{27}{32}\bar{a}(s)|_{s=2(1,x^2,x^3)}. \tag{2.27}$$

We checked explicitly that this relation holds in many examples. If proven in general, (2.27) will establish the equivalence $a_{CFT} = a_{MSY}$ in a somewhat more direct way than the approach of [28] rederived in section 2.3.

For the rest of this paper, we shall assume that (2.27) holds, and examine its implications. It is convenient to reinstate the s^1-dependence of the quantities we defined earlier. For example,

$$\bar{L}^I(s) \equiv \frac{\langle v_{I-1}, v_I, v_{I+1}\rangle}{\langle s, v_{I-1}, v_I\rangle\langle s, v_I, v_{I+1}\rangle}, \qquad \bar{S}(s) \equiv \frac{1}{s^1}\sum_I \bar{L}^I(s), \qquad h^I_{MSY}(s) \equiv \frac{\bar{L}^I(s)}{\bar{S}(s)}. \tag{2.28}$$

Note that $h^I(s)$ satisfies

$$h^I(s) v_I^j = s^j. \tag{2.29}$$

For $\bar{h}^I$, this holds due to the toric relation (2.7), while for h^I_{MSY} it has a geometric explanation, which we review in appendix B. Differentiating, we find

$$\frac{\partial h^I}{\partial s^i} v_I^j = \delta_i^j. \tag{2.30}$$

Thus $(\partial h^I/\partial s^i)$ satisfy the same relation as F_i^I in (2.7). We therefore define the 'canonical' flavor charge as

$$\hat{F}_i^I \equiv \left.\frac{\partial h^I}{\partial s^i}\right|_{s=s_*} \qquad (i=1,2,3), \tag{2.31}$$

where s_* denotes the value of s which maximizes the a-function. We will show in the next section that this is precisely the flavor charge captured by supergravity.

An important feature of the canonical flavor charge is that it makes $\hat{C}_{Ria} \equiv s_*^j \hat{F}_j^I \hat{F}_i^J \times Q_a^K C_{IJK}$ vanish. Suppose we work in the 'canonical frame', that is, we substitute $\hat{F}_i^I$ for F_i^I in (2.22) and proceed. Since h^I is a homogeneous function of s of degree 1, we can always write $h^I(s) = s^i \frac{\partial h^I}{\partial s^i}$. In the canonical frame, this implies that $\bar{t}^a(s_*) = 0$. Next, by differentiating (2.25) in the canonical frame, and recalling (2.31),

$$\frac{\partial h^I}{\partial s^i} = Q_a^I \frac{\partial \bar{t}^a}{\partial s^i} + \hat{F}_i^I \qquad \Longrightarrow \qquad \left.\frac{\partial \bar{t}^a}{\partial s}\right|_{s_*} = 0. \tag{2.32}$$

Now, combining $\bar{t}^a(s_*) = 0 = \frac{\partial \bar{t}^a}{\partial s^i}|_{s_*}$ with $\bar{t}^a(s) = -m^{ab}(s) n_b(s)$, we find that

$$\left.\frac{\partial n_a}{\partial s^i}\right|_{s_*} = C_{aij}\, s_*^j = C_{Rai} = 0. \tag{2.33}$$

This demonstrates the decoupling property among the global charges.

3. Comparison with Supergravity

In this section we compare our main results from the previous section with the supergravity computation. First, we work out the KK reduction at the linearized level. It was already done in [30] where a covariant action in ten dimensions was assumed. To avoid the usual difficulty with the self-dual five form of IIB supergravity, we follow the common path [36, 37] of using only the equations of motion.

Second, we compare the flavor charges between field theory and supergravity. The agreement is perfect. We emphasize that both field theory and supergravity pick out a unique value of flavor charge and the mixing ambiguity is resolved.

Finally, we would like to compare C_{IJK} of field theory (2.9) with supergravity by extending the KK reduction to the cubic order. This has been carried out in a very recent paper [40]. In the last subsection of this paper, we check the relation $\tau_{IJ} = -3C_{RIJ}$ [33] using the results of [30, 40] and find complete agreement.

3.1 Massless vectors from linearized equations

We shall follow the conventions of [37]. The IIB supergravity equations of motion relevant to our analysis are

$$R_{mn} = \frac{4}{4!} F_{mi_1i_2i_3i_4} F_n{}^{i_1i_2i_3i_4}, \qquad F = *F, \qquad dF = 0. \tag{3.1}$$

In units in which the 'radius' $l = (4\pi^4 g_s N/\mathrm{Vol}(Y))^{1/4} l_s$ is set to be unity, the background solution with N units of F-flux is

$$ds^2 = ds^2_{AdS} + ds^2_Y \qquad \text{and} \qquad F = \mathrm{vol}_{AdS} + \mathrm{vol}_Y. \tag{3.2}$$

The metric is normalized such that $R_{\mu\nu} = -4g_{\mu\nu}$ for AdS_5 and $R_{\alpha\beta} = +4g_{\alpha\beta}$ for Y. We shall now perturb around the background solution and obtain equations of motion for massless vector gauge fields up to linear order.

The gauge fields for baryon symmetries arise from fluctuations of the RR five-form field strength,

$$\delta F = F^a \wedge \omega_a - *F^a \wedge *\omega_a, \tag{3.3}$$

around the background (3.2). The second term ensures that the self-duality constraint $F = *F$ is satisfied. Here, the Hodge duals are factorized to AdS_5 and Y, respectively. At the linearized level, no other perturbation is needed.

The gauge fields for flavor symmetries arise from fluctuations along the isometries. We take the following ansatz for the fluctuations:

$$ds^2 = ds^2_{AdS} + g_{\alpha\beta}(dy^\alpha + K_i^\alpha A^i)(dy^\beta + K_j^\beta A^j), \tag{3.4}$$

$$F = \mathrm{vol}_{AdS} + \mathrm{vol}_Y + dC, \qquad C = \frac{1}{8}(B^i \wedge *dK_i + *dB^i \wedge K_i). \tag{3.5}$$

The metric part of the ansatz is the standard one in KK reduction. The vector B^i from the RR five-form field-strength must be turned on also because A^i and B^i mix already at linearized order [36]. As the ansatz for F is written in terms of the potential C, the Bianchi identity holds automatically. Again, the Hodge duals are factorized to AdS_5 and Y, respectively.

The mixed components of the Einstein equation and the self-duality equation give, respectively,

$$(\Box - 8)A^i = (\Box + 8)B^i \qquad \text{and} \qquad (\Box - 8)B^i = 8A^i, \tag{3.6}$$

where we defined $\Box \equiv (*d*d)_{AdS}$. We also used the fact that $d*K_i = 0$, $d*dK_i = 8*K_i$ on Y, which follows from the Killing equation $\nabla_\alpha K_\beta + \nabla_\beta K_\alpha = 0$ and $R_{\alpha\beta} = 4g_{\alpha\beta}$. We can easily diagonalize the two equations to obtain the mass eigenstates:

$$\Box(A^i + B^i) = 24(A^i + B^i), \qquad \Box(A^i - 2B^i) = 0. \tag{3.7}$$

To keep the massless fields only, we set $B^i = -A^i$.

Now, we can read off the gauge kinetic term of the massless gauge fields from the AdS_5 components of the field equations (3.1). They yield via AdS/CFT the coefficients τ_{IJ} of the two-point correlators for conserved global currents J_I in gauge theory. The result is to be compared with [30]. A precise comparison, however, requires normalization of the gauge fields, which is related to the normalization of the charges on the gauge theory side. So, we shall first discuss how to find the correct normalization.

3.2 Charges

As stated in (2.6), a natural normalization for the baryon charges is

$$B_a\left[\Sigma^I\right] = \int_{\Sigma^I} \omega_a = Q_a^I, \tag{3.8}$$

where $\{\omega_a\}$ form an integral basis of $H^3(Y, \mathbb{R})$. The KK analysis of the previous subsection suggests that the flavor charges can be measured with the replacement of ω_a by $*dK_i$ modulo an arbitrary multiplicative constants. The correct normalization turns out to be

$$F_i^I = \frac{2\pi}{V} \int_{\Sigma^I} (*dK_i)/8 \qquad\qquad (i = 1, 2, 3), \tag{3.9}$$

where V denotes $\mathrm{Vol}(Y)$. As a first check, note that the R-charge is given by

$$R^I = \frac{2}{3} b^i F_i^I = \frac{\pi}{6V} \int_{\Sigma^I} *dK_R = \frac{\pi}{3V} \mathrm{Vol}(\Sigma^I), \tag{3.10}$$

in agreement with the well-known result in the literature [35]. Note that we are abusing the notations a bit and use K_i to denote both the Killing vector and its dual one-form. In the last step of (3.10), we used the local $U(1)_R$ fibration description of the SE manifold Y (see also (2.3)):

$$ds_Y^2 = (e^0)^2 + ds_B^2, \;\; e^0 \equiv \frac{1}{3} d\psi + \sigma, \qquad K_R = 3 \frac{\partial}{\partial \psi}, \tag{3.11}$$

$$R_{\mu\nu}^{(B)} = 6 g_{\mu\nu}^{(B)}, \qquad d\sigma = 2J_B, \qquad \mathrm{vol}_\Sigma = e^0 \wedge J_B. \tag{3.12}$$

It is instructive to compare (3.9) with known results. On the supergravity side, generalizing the analysis for the R-charge in [35], the authors of [30] showed that, for *non-R* flavor charges,

$$F_i^I = -\frac{\pi}{V}\int_{\Sigma^I}(i_{K_i}\sigma)\mathrm{vol}_\Sigma = -\frac{2\pi}{V}\int_{\Sigma^I} y_i \,\mathrm{vol}_\Sigma = -\frac{1}{V}\frac{\partial V}{\partial v_I^i} \qquad (i=2,3), \tag{3.13}$$

where in the last expression, the volume V is regarded as a function of the toric data v_I^i. On the other hand, as we reviewed in the last section the field theory result is

$$F_i^I = \frac{1}{2}\frac{\partial}{\partial x^i}h^I_{MSY}(\vec{x}) \qquad (i=2,3). \tag{3.14}$$

We now show that all three expressions for the flavor charges (3.9), (3.13) and (3.14) are in fact the same. To see (3.14) is the same as the last expression in (3.13), we note that

$$h^I_{MSY} = \frac{2L^I}{S}, \qquad \left.\frac{\partial V}{\partial x^i}\right|_{x_*} = 0, \qquad \frac{\partial S}{\partial v_I^i} = -\frac{\partial L^I}{\partial x^i}, \tag{3.15}$$

where x_* denotes the value of $\vec{x}$ that minimizes S which is proportional to $V = \mathrm{Vol}(Y)$. The last identity in (3.15) holds for arbitrary values of $\vec{x}$, as can be checked by explicit computation.

To see that the first expression in (3.13) is the same as (3.9), it suffices to show the equality:

$$\int_{\Sigma^I} *_5 dK_i = -4\int_{\Sigma^I}(i_{K_i}\sigma)\mathrm{vol}_\Sigma. \tag{3.16}$$

This can be proven using (3.11), (3.12). The one-form dual to the flavor Killing vector $K_i = \partial/\partial\phi^i$ $(i=2,3)$ can be decomposed into the base B and the local $U(1)_R$ fiber:

$$K_i = \bar{K}_i + (i_{K_i}\sigma)e^0 \qquad \text{such that} \qquad dK_i = d\bar{K}_i + 2(i_{K_i}\sigma)J_B - 2(i_{K_i}J_B)e^0. \tag{3.17}$$

Here, the relation $\mathcal{L}_{K_i}\sigma \equiv d(i_{K_i}\sigma) + i_{K_i}(d\sigma) = 0$ was used. Splitting the three-cycle Σ^I into the $U(1)_R$ fiber and a 2-cycle B^I in the base B,

$$\int_{\Sigma^I} *_5 dK_i = \int e^0 \int_{B^I} *_4 d\bar{K}_i + 2\int_{\Sigma^I}(i_{K_i}\sigma)\mathrm{vol}_\Sigma. \tag{3.18}$$

The final step of the proof follows from the identity:

$$d\bar{K}_i + *_4 d\bar{K}_i = -6(i_{K_i}\sigma)J_B. \tag{3.19}$$

The left-hand side of (3.19) is manifestly a self-dual $(1,1)$ form, so it must be proportional to the Kähler form J_B. To see if (3.19) is consistent, take an exterior derivative to (3.19). We find that $d *_4 d\bar{K}_i = 12 *_4 \bar{K}_i$ from the left-hand side is indeed equal to

$$-6\,d(i_{K_i}\sigma)\wedge J_B = 12(i_{K_i}J_B)\wedge J_B = 12 *_4 \bar{K}_i$$

from the right-hand side. This still leaves a room for a term proportional to the Kähler form J_B on the right-hand side of (3.19). To show that such a term does not appear, let us now integrate (3.19) over the base B. The left-hand side vanishes by integration parts and $dJ = 0$, while

$$\int_B (i_{K_i}\sigma) \quad \propto \quad \int_Y (i_{K_i}\sigma) \quad \propto \quad \frac{\partial V}{\partial b^i} = 0, \tag{3.20}$$

as a result of volume-minimization [9, 30].

3.3 Gauge kinetic coefficient τ_{IJ} revisited

With the normalization for the flavor charges fixed, from the KK reduction analysis in section 3.1, we can compute the gauge field kinetic term coefficient τ_{IJ} and compare them with [30]. To do so in uniform manner along with the flavor charges (3.9), we rescale the harmonic three-forms by $2\pi/V$ relative to (3.8), viz.

$$\frac{2\pi}{V}\int_{\Sigma^I}\omega_a = Q^I_a. \tag{3.21}$$

Then, the expressions for τ_{IJ} are

$$\tau_{ab} = \frac{16\pi^3}{V^2}\int_Y \omega_a \wedge *\omega_b, \qquad \tau_{ai} = 0, \qquad \tau_{ij} = \frac{3\pi^3}{V^2}\int_Y K_i \wedge *K_j. \tag{3.22}$$

The baryon components τ_{ab} are precisely the same as in [30]. As for the flavor components, the coefficient of gravi-photon (R-symmetry) is

$$\tau_{RR} = \left(\frac{2}{3}\right)^2 b^i b^j \tau_{ij} = \frac{3\pi^3}{V^2}\left(\frac{2}{3}\right)^2 \int_{Y_5} K_R \wedge *K_R = \frac{4\pi^3}{3V} = \frac{16}{3}a, \tag{3.23}$$

in agreement with [30]. For the other flavor symmetries, the expression from [30] looks slightly different:

$$\tau_{ij} = \frac{12\pi^3}{V^2}\int_Y (i_{K_i}\sigma)(i_{K_j}\sigma)\mathrm{vol}_Y \qquad (i,j = 2,3). \tag{3.24}$$

It agrees with (3.22) if and only if

$$\int_{Y_5} K_i \wedge *K_j = 4\int_Y (i_{K_i}\sigma)(i_{K_j}\sigma)\mathrm{vol}_Y \qquad (i,j = 2,3). \tag{3.25}$$

This identity was stated in [30] without proof. We note that it can be verified using (3.19), and other relations we used in section 3.2. See appendix C for details.

3.4 Chern-Simons coupling C_{IJK}

The Chern-Simons coupling C_{IJK} is obtainable in KK reduction by using the ansatz of subsection 3.1 and computing the fluctuation up to cubic order along the line of [37–39]. While this work was in progress, ref. [40] appeared, where the full computation was performed using a slightly different approach. The difference is that our ansatz manifestly satisfy $dF = 0$ but the self-duality equation is non-trivial, while an alternative ansatz was used in [40], where F is manifestly self-dual but not necessarily closed.

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A central step in [40] was to combine the baryon symmetries and flavor symmetries together into some three-forms ω_I such that

$$\int_{\Sigma^I} \omega_J = \delta^I_J. \tag{3.26}$$

Comparing with our charge normalizations (3.9), (3.21) and the toric relation (2.7), we find that

$$\omega_I = \frac{2\pi}{V}(u_I^a \omega_a + v_I^i * dK_i/8). \tag{3.27}$$

We can use it to re-express the result of [40] in a more convenient form:

$$\begin{aligned} C_{ijk} &= \frac{3\pi^3}{8V^2}\int_Y K_i \wedge dK_j \wedge dK_k, \\ C_{ija} &= \frac{2\pi^3}{V^2}\int_Y *(K_i dK_j)\wedge \omega_a, \\ C_{iab} &= \frac{8\pi^3}{V^2}\int_Y \omega_a \wedge i_{K_i}\omega_b. \end{aligned} \tag{3.28}$$

As a consistency check, we compute the a-function, which is proportional to C_{RRR}, and obtain the expected result:

$$\begin{aligned} a = \frac{9}{32}C_{ijk}b^i b^j b^k \left(\frac{2}{3}\right)^3 &= \frac{\pi^3}{32V^2}\int_Y K_R \wedge dK_R \wedge dK_R \\ &= \frac{\pi^3}{32V^2}\int_Y e^0 \wedge (2J_B)\wedge(2J_B) = \frac{\pi^3}{4V}. \end{aligned} \tag{3.29}$$

3.5 $\tau_{IJ} = -3C_{RIJ}$ relations

Utilizing the supergravity expressions for the gauge kinetic coefficients (3.22) and the Chern-Simons coefficients (3.28), we can now demonstrate the relation suggested in [33] between the two-point correlators and the triangle 't Hooft anomalies involving conserved currents in the gauge theory:

$$\tau_{IJ} = -3\,\mathrm{Tr}\,R\,F_I\,F_J \equiv -3C_{RIJ}. \tag{3.30}$$

Here, F_I include both baryon and non-R flavor charges.

First, $\tau_{ab} = -3C_{Rab}$ follows from the fact that, in the local $U(1)_R$ fibration description of Y given in (3.11), (3.12), $\omega_a = e^0\eta_a$ for some *anti*-self-dual two-form η_a on B [30]. Next, $\tau_{ia} = 0$ implies that C_{Ria} must also vanish. It is indeed so because $K_R = e^0$, $\omega_a = e^0\eta_a$ as mentioned above, and ω_a is harmonic. This also agrees with the field theory computation (2.33). The last relation $\tau_{ij} = -3C_{Rij}$ amounts to

$$\int K_R \wedge dK_i \wedge dK_j = -4\int K_i \wedge *K_j. \tag{3.31}$$

This simply follows from (3.25), as explained in appendix C.

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A. $\mathrm{Tr}(B^3) = 0$

We prove that $C_{IJK}B^IB^JB^K = 0$ for any linear combination of baryon symmetries. The proof consists of a combination of our formula $C_{IJK} = |\langle v_I, v_J, v_K\rangle|/2$, the toric relation $B^Iv_I = 0$, and some combinatoric manipulations. More concretely, we show that

$$\begin{aligned} 0 &= \frac{1}{2}\sum_{(J,K)}\left[\sum_I \langle v_I, v_J, v_K\rangle B^IB^JB^K \times (d-2(K-J))\right] \\ &= \sum_{(J,K)}\left[\sum_I (-1)^{(I,J,K)} C_{IJK}B^IB^JB^K \times (d-2(K-J))\right] \\ &= \frac{d}{6}\sum_{I,J,K} C_{IJK}B^IB^JB^K. \end{aligned} \tag{A.1}$$

The notations require some clarification. The (J, K) sum runs over all possible pairs with $0 < K - J \le d/2 \pmod d$. The I sum then runs over all vertices. The first line is a trivial consequence of $B^Iv_I = 0$. The second line simply says that C_{IJK} is equal to $\langle v_I, v_J, v_K\rangle/2$ up to a sign depending on whether I lies on the long(+) or short(-) path between J and K. The weight factor $d - 2(K - J)$ ensures that if we choose some fixed triangle (I, J, K) and collect all terms proportional to C_{IJK} from the second line, the net coefficient always turns out to be d, independent of the choice of the triangle.

Let us check the last statement. Let l_1, l_2, l_3 be the number of edges between (I, J), (J, K) and (K, I) respectively, so that $l_1 + l_2 + l_3 = d$. Without loss of generality, we may assume that $l_1 \le l_2 \le l_3$. We collect the terms in two separate cases:

1. $l_3 \le d/2$: The sign is positive for all three contributions from the second line of (A.1). The net coefficient is $(d - 2l_1) + (d - 2l_2) + (d - 2l_3) = d$.

2. $l_3 > d/2$: The sign is negative in one of the three contributions from the second line of (A.1). The net coefficient is $(d - 2l_1) + (d - 2l_2) - (d - 2(d - l_3)) = d$.

This completes the proof.

B. Equality of a_{CFT} and a_{MSY}

We prove the lemma (2.20):

$$c_I \equiv C_{IJK}L^JL^K = 3S + \langle r_I, u\rangle. \tag{B.1}$$

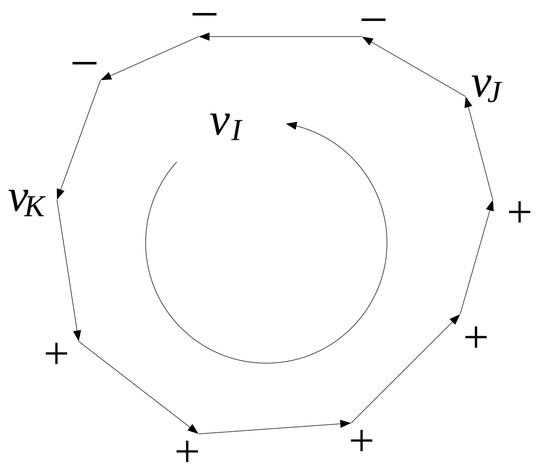

Figure 3: The sign assignment in the second line of (A.1).

As explained in section 2.3, this lemma is sufficient to establish the equality between a_{CFT} and a_{MSY}. The main idea for the proof is the same as in the original one [28], but our formula $C_{IJK} = |\langle v_I, v_J, v_K\rangle|/2$ simplifies the computation involved considerably.

The definition of w_I, r_I, etc. are the same as in section 2.3. In what follows, we will need the following identity [28]:

$$L^I r_I = \frac{w_{I-1}}{A_{I-1}} - \frac{w_I}{A_I}, \quad \Longrightarrow \quad \sum_I L^I r_I = 0 \quad \text{or} \quad \sum_I L^I v_I = (1, x^2, x^3) \sum_I L^I. \tag{B.2}$$

Geometrically, the last equation follows from integrating the 'gradient of a constant function' over the polyhedral cone Δ and applying Stokes' theorem; see (2.91) of [9].

Getting back to the lemma, we write c_1 as

$$c_1 = \sum_{2\to d} \langle v_1, v_J, v_K\rangle L^J L^K. \tag{B.3}$$

Here the notation $(2 \to d)$ means that the sum is taken over $2 \le J < K \le d$. In the following, we will use notations like $(2 \to 1)$, which means the range $2 \le J < K \le d+1$ with $v_{d+1} \equiv v_1$.

As in [28], we first compute the difference between two adjacent c_I's. Using the relation $\langle v_I, v_J, v_K\rangle = \langle r_I, r_J\rangle + \langle r_J, r_K\rangle + \langle r_K, r_I\rangle$, we find, for example,

$$c_2 - c_1 = \langle w_1, u_1\rangle, \qquad u_1 \equiv \sum_{2\to 1} (r_J - r_K) L^J L^K - 2S\frac{w_1}{A_1}. \tag{B.4}$$

The second term in the definition of u_1 does not affect the value of $c_2 - c_1$. We include it (and similar terms for all u_I) to make all the u_I's the same $(u_1 = u_2 = \cdots = u_d \equiv u)$:

$$\begin{aligned} u_2 - u_1 &= -2\sum_{3\to 1} (r_2 - r_K) L^2 L^K - 2S\left(\frac{w_2}{A_2} - \frac{w_1}{A_1}\right) \\ &= -2\left[r_2 L^2 (S - L^2) + r_2 (L^2)^2\right] - 2S\left(\frac{w_2}{A_2} - \frac{w_1}{A_1}\right) = 0, \end{aligned} \tag{B.5}$$

where we used (B.2). This implies that $c_I - \langle r_I, u\rangle$ is independent of the index I. Performing the subtraction and using (B.2) once again, we find

$$c_1 - \langle r_1, u\rangle = 2S + \sum_{2\to d}\langle r_J, r_K\rangle L^J L^K \equiv 2S + T. \tag{B.6}$$

Finally, we show that $T = S$ by mathematical induction. To begin with, we note that for $d = 3$,

$$T = \langle r_2, r_3\rangle L^2 L^3 = A_2 \times \frac{\langle w_1, w_2\rangle}{A_1 A_2} \times \frac{\langle w_2, w_3\rangle}{A_2 A_3} = \frac{\langle w_1, w_2\rangle}{A_1 A_2} + \frac{\langle w_2, w_3\rangle}{A_2 A_3} + \frac{\langle w_3, w_1\rangle}{A_3 A_1} = S, \tag{B.7}$$

where we used the fact that, when $d = 3$, $\langle w_1, w_2\rangle = \langle w_2, w_3\rangle = \langle w_3, w_1\rangle = A_1 + A_2 + A_3$. Now, assume that $T = S$ holds for a toric diagram with d vertices. As we add another vertex v_{d+1}, most of the terms in S and T remain unchanged. The only differences are

$$\tilde{S} - S = \tilde{L}_d + \tilde{L}_{d+1} + \tilde{L}_1 - (L_d + L_1), \tag{B.8}$$
$$\tilde{T} - T = \langle r_d, r_1\rangle \tilde{L}^d \tilde{L}^1 + \langle r_d, r_{d+1}\rangle \tilde{L}^d \tilde{L}^{d+1} + \langle r_{d+1}, r_1\rangle \tilde{L}^{d+1}\tilde{L}^1 - \langle r_d, r_1\rangle L^d L^1, \tag{B.9}$$

where we distinguished the objects for the new diagram by adding tilde above them. Using the identity again (B.2), we obtain

$$\langle r_d, r_1\rangle L^d L^1 = L_d + L_1 - \frac{\langle w_{d-1}, w_1\rangle}{A_{d-1}A_1}, \tag{B.10}$$
$$\langle r_d, r_{d+1}\rangle \tilde{L}^d \tilde{L}^{d+1} = \tilde{L}_d + \tilde{L}_{d+1} - \frac{\langle w_{d-1}, \tilde{w}_{d+1}\rangle}{A_{d-1}\tilde{A}_{d+1}}, \tag{B.11}$$
$$\langle r_{d+1}, r_1\rangle \tilde{L}^{d+1} \tilde{L}^1 = \tilde{L}_{d+1} + \tilde{L}_1 - \frac{\langle \tilde{w}_d, w_1\rangle}{\tilde{A}_d A_1}, \tag{B.12}$$
$$\langle r_d, r_1\rangle \tilde{L}^d \tilde{L}^1 = -\tilde{L}_{d+1} + \frac{\langle w_{d-1}, \tilde{w}_{d+1}\rangle}{A_{d-1}\tilde{A}_{d+1}} + \frac{\langle \tilde{w}_d, w_1\rangle}{\tilde{A}_d A_1} - \frac{\langle w_{d-1}, w_1\rangle}{A_{d-1}A_1}. \tag{B.13}$$

Therefore, $T = S$ implies $\tilde{T} = \tilde{S}$. This completes the proof.

C. Some identities

In this appendix, we prove two identities that we needed in section 3 to establish the relation between τ_{ij} and C_{ijk}. Recall that the one-form dual to the Killing vector K_i is decomposed under the local $U(1)_R$ fibration description of Y (3.11), (3.12) as

$$K_i = \bar{K}_i + (i_{K_i}\sigma)e^0. \tag{C.1}$$

The integral appearing in τ_{ij} splits accordingly:

$$\frac{1}{2\pi}\int_Y K_i \wedge *_5 K_j = \int_B \bar{K}_i \wedge *_4 \bar{K}_j + \int_B (i_{K_i}\sigma)(i_{K_j}\sigma)\mathrm{vol}_B \equiv A_{ij} + B_{ij}. \tag{C.2}$$

The first identity (3.25) follows from a straightforward computation:

$$\begin{aligned} A_{ij} &= \int_B (i_{K_i} J_B) \wedge *_4 (i_{K_j} J_B) = -\frac{1}{2}\int_B d(i_{K_i}\sigma) \wedge *_4 (i_{K_j} J_B) \\ &= \frac{1}{2}\int_B (i_{K_i}\sigma) d *_4 (i_{K_j} J_B) = -\frac{1}{2}\int_B (i_{K_i}\sigma) d(\bar{K}_j \wedge J_B) \\ &= -\frac{1}{2}\int_B (i_{K_i}\sigma)\left[\frac{1}{2}(d\bar{K}_j + *_4 d\bar{K}_j) + \frac{1}{2}(d\bar{K}_j - *_4 d\bar{K}_j)\right] \wedge J_B \\ &= 3\int_B (i_{K_i}\sigma)(i_{K_j}\sigma)\frac{1}{2} J_B \wedge J_B \quad = \quad 3B_{ij}. \end{aligned} \tag{C.3}$$

We used (3.19) in going from the third to the last line. The second identity (3.31) follows, since

$$\begin{aligned} \frac{1}{2\pi}\int_Y K_R \wedge dK_i \wedge dK_j &= \int_B (d\bar{K}_i + 2(i_{K_i}\sigma)J_B) \wedge (d\bar{K}_j + 2(i_{K_j}\sigma)J_B) \\ &= -8A_{ij} + 8B_{ij} = -16B_{ij} = -4\left[\frac{1}{2\pi}\int_{Y_5} K_i \wedge *_5 K_j\right]. \end{aligned} \tag{C.4}$$

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